Combinatorial Properties of Billiards on an Equilateral Triangle

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Abstract

We explore periodic orbits on an equilateral triangular billiard table. We prove that there is exactly one periodic orbit with odd period. For any positive integer n, there exist $\sum_{d|n} \mu(d) P\left(\frac{n}{d}\right)$ periodic orbits of period $2n$, where $\mu(d)$ is the Möbius transformation function and $P(n) = \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n+2}{3} \rfloor$. We count periodic orbits by introducing a new type of integer partition.

1 Introduction

Consider a triangular billiards table. Sending a ball into motion on this table causes it to bounce about the table along a path completely determined by its initial position, angle of departure, and initial speed. If we assume negligible friction and observe the path for an arbitrarily long period of time we can ignore the speed parameter and simplify the situation by starting our observations at the point of the first bounce.

We assume that the action of a ball bouncing off a wall follows the standard rule of reflection: *the angle of incidence equals the angle of reflection.* This rule cannot apply when a ball strikes the corner, so we follow convention and decide that a path terminates upon reaching a corner. Before proceeding, we must define the terminology that will be used for the remainder of the paper.

Definition 1 *An orbit is the path that the ball follows. A terminal orbit is an orbit that terminates at a vertex. A periodic orbit of period n is an orbit that retraces itself after n bounces. An infinite orbit is any orbit that is neither terminal or periodic.*

These problems are classically known as "billiards problems." Tabachnikov's work "Billiards" [5] provides a thorough compilation of known results regarding the existence of periodic orbits on surfaces with various bounding curves. For example, it has been proven that any acute, isosceles, or right triangle admits a periodic orbit. Certain types of obtuse triangles have also been proven to admit periodic orbits, but the existence of a periodic orbit on any triangle remains an open question [2], [3], [6].

The existence of a periodic orbit on an equilateral triangle was proven by Fagnano in 1745. In 1986, Howard Masur proved that any rational polygon, that is a polygon all of whose interior angles are rational multiples of π, yields an infinite number of periodic orbits. Specifically, the equilateral triangle yields an infinite number of periodic orbits [3]. Masur does not, however, show how to construct these orbits or count how many exist of a given period.

In section 2 we review the existence of Fagnano's periodic orbit on the equilateral triangle: a period 3 orbit and a larger collection of related period 6 orbits. Section 3 reveals a method for locating periodic orbits via reflections, and we prove that the period 3 orbit is the only periodic orbit with odd period. In Section 4 we impose a coordinate system which facilitates finding vectors representing period $2n$ orbits. In section 5, we use number theory and combinatorics to showing that there are $P(n) = \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n+2}{3} \rfloor$ coordinate pairs (i.e. vectors) which represent period 2n orbits for any natural number n.

A problem arises, however, as this formula counts orbits that may be little more than repetitions of lower-period orbits. For example, the single period 8 orbit counted by $P(4)$ is actually two repetitions of a period 4 orbit counted as a single unit. In section 6, we set out to separate the duplicate-free orbits (those which are not repetitions) from the orbits containing duplicates and enumerate each class. We prove that there exists at least one duplicate-free period $2n$ orbit, granted that $n \neq 1, 4, 6, 10$. Furthermore, if we let $\mu(d)$ be the Möbius function given by

$$\mu(d) = \begin{cases} 1, & d = 1, \\ (-1)^r, & d = p_1 p_2 \cdots p_r \text{ with } p_i\text{'s distinct primes,} \\ 0, & \text{otherwise} \end{cases}$$

then there are $\sum_{d|n} \mu(d) P\left(\frac{n}{d}\right)$ duplicate-free period $2n$ orbits, where d ranges over all divisors of n.

It must be made clear before proceeding that the periodic orbits discussed (save for the unique period 3 orbit) are in truth infinite *families* of orbits. Periodic orbits commonly appear in groups sharing an initial angle whose initial points form a connected subset of the initial side. For simplicity, these families will be treated as a single periodic orbit, distinguished by its initial angle. Constructing a periodic orbit revolves primarily around finding the proper initial angle, since only countably many initial points yield terminal orbits for a given angle.

2 Fagnano's Orbit

Theorem 2 *An equilateral triangle admits a periodic orbit.*

Proof. The simplest orbit is the period 3 "orthoptic" [2] or "Fagnano" [5] orbit. Starting at any midpoint at a 60° angle, the ball proceeds to bounce on the midpoint of each side, forming the period 3 orbit shown in Figure 1.

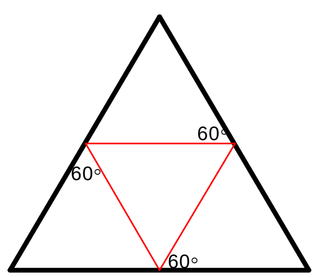

Figure 1: The period 3 orbit.

If the ball does not begin at the midpoint, a period 6 orbit arises, as seen in Figure 2.

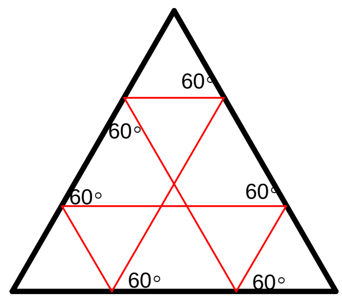

Figure 2: A period 6 orbit.

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3 Unfolding Orbits and Odd-Period Orbits

Elementary geometry proves reflecting the triangle about a side is equivalent to bouncing the ball on the side. The "unfolding" action makes orbits much easier to handle.

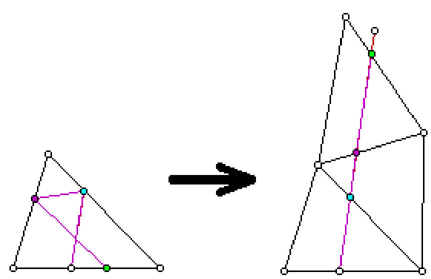


Figure 3: An unfolding.

An orbit is periodic if and only if the ball returns to its initial point at its initial angle. In the unfolded triangles, any line that connects a point to its image is a periodic orbit if the image lies on an edge parallel to the initial edge. To demonstrate, examine the unfolded versions of the period 6 and period 10 orbits in Figure 4. The converse holds save for a single case. Unfolding the period 3 orbit shows that it does not satisfy the parallel edge requirement. If the trajectory is continued through three more reflections (six in all), one sees that the orbit does eventually reach an image of the original point that lies on a parallel edge, however.

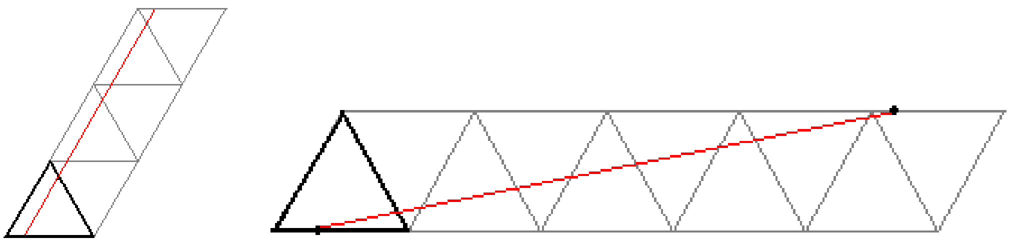


Figure 4: The unfolded period 6 and period 10 orbits.

The symmetry of the equilateral triangle allows us to tessellate the plane by reflecting in its edges and their images. The ability to generate a tessellation in this way is unique to equilateral triangles, since performing multiple reflections in other triangles can lead to overlap. We are interested in the lines in the tessellation and not the triangles themselves. For clarity, an edge is the line segment between two adjacent vertices.

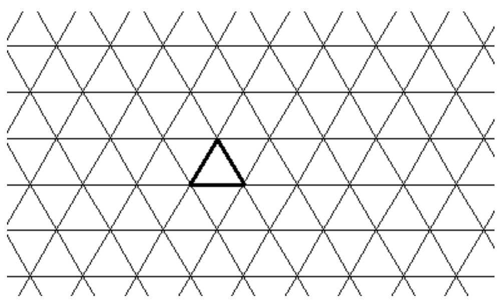


Figure 5: A tessellation of the plane with equilateral triangles.

Any vector drawn on this framework of lines represents an orbit on the equilateral triangle. The task of finding and classifying periodic orbits on the equilateral triangle reduces to finding and classifying vectors that represent periodic orbits. For simplicity, all vectors will have initial points on horizontal edges.

Definition 3 *Let Γ be an orbit with initial point P on edge e and terminal point on line segment m. Then m lies in the bounce circle of radius n centered at P ($n \in \mathbb{N}$) if Γ crosses n lines excluding e. The edge e is the bounce circle of radius 0, and the bounce circle of radius ∞ is defined to be the line containing e, minus e itself.*

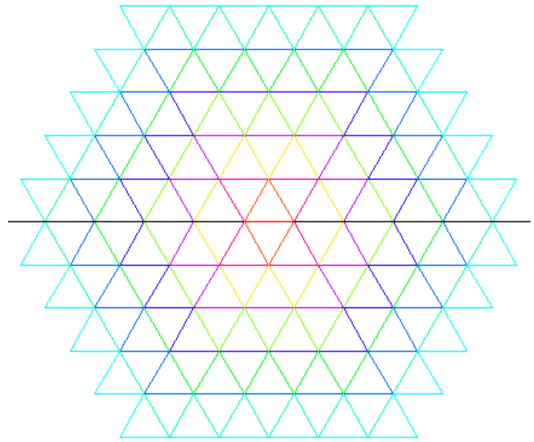


Figure 6: Bounce circles of radii 0 - 10, ∞ (coded by color).

Clearly every edge is in exactly one bounce circle. Note that every horizontal line is contained in a bounce circle with even or infinite radius. Further, notice that the bounce circles of odd radius are composed entirely of left- and right-leaning diagonals.

Theorem 4 *The equilateral triangle admits exactly one periodic orbit of odd period, the period 3 orthoptic orbit.*

Proof. Let Γ be the vector representation of a periodic orbit with initial point P and terminal point Q on a bounce circle of odd radius. Let θ be the angle formed by Γ and the edge containing P; let α be the angle formed by Γ and the edge containing Q. Since Γ is periodic, $\theta = \alpha$. Finding values for θ and α depends on whether Q lies on a left- or right-leaning diagonal.

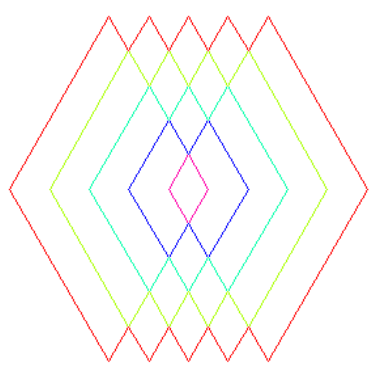


Figure 7: Bounce circles with radii 1,3,5,7 and 9.

Case 1: If Q lies on a right-leaning diagonal, then $\theta = \alpha = 30°$. However, $\theta = 30°$ does not yield any crossings with odd bounce circles on right-leaning diagonals, ruling out this case.

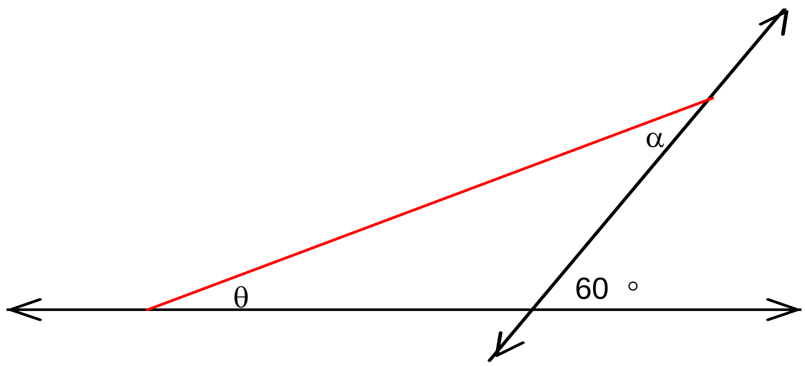


Figure 8: Point Q on a right-leaning diagonal.

Case 2: If Q lies on a left-leaning diagonal, then $\theta = \alpha = 60°$. As Theorem 2 shows, $\theta = 60°$ yields the period 3 or period 6 orbit.

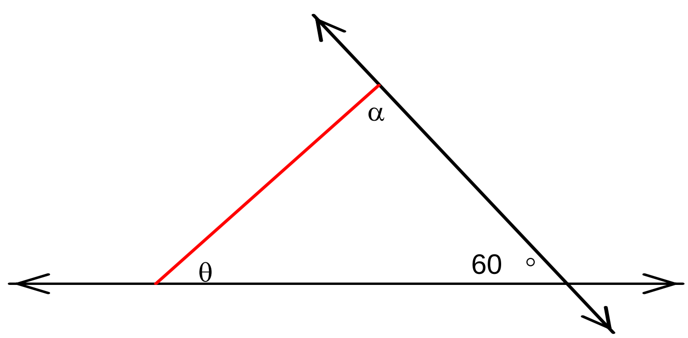


Figure 9: Point Q on a left-leaning diagonal.

$\blacksquare$

Since the period 3 orbit is such a special case, we treat it as a degenerate period 6 orbit from this point forward. Thus all orbits will be spoken of has having period $2n$ for a positive integer n.

4 A Coordinate System

There is a natural coordinate system on the equilateral triangle tessellation. Choose initial point P to be the origin, let the x-axis remain horizontal, and let the y-axis be parallel to the right-leaning diagonals. Setting the original triangle as a unit triangle, we obtain the coordinate system shown below.

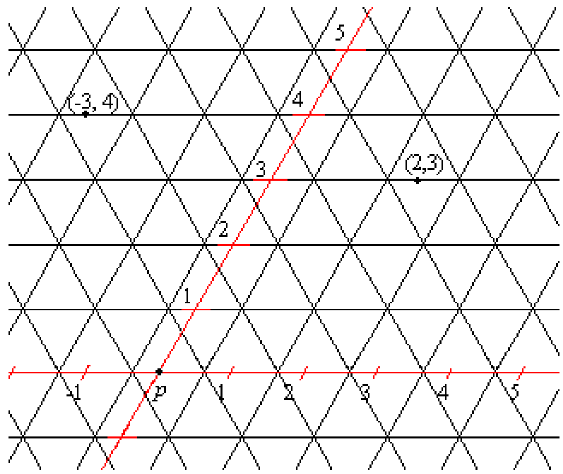


Figure 10: The rhombic coordinate system.

Elementary matrix algebra yields the following change of basis matrices:

From Euclidean to Rhombic:	From Rhombic to Euclidean:
$\begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{3}} \end{pmatrix}$	$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$

All ordered pairs from this point forward are written as points or vectors in this rhombic basis. Given a vector $\Gamma = (x, y)$, $y > 0$, one can use the following formulas to compute the geometric length L and initial angle θ.

$$L((x,y)) = \sqrt{x^2 + xy + y^2} \tag{1}$$

$$\theta((x,y)) = \arctan\left(\frac{y\sqrt{3}}{2x+y}\right) \tag{2}$$

There is a surprisingly simple method to determine if a given vector represents a periodic orbit.

Theorem 5 *An orbit vector (x, y) is periodic if and only if x and y are integers such that $x \equiv y \pmod 3$.*

Proof. Vector (x, y) is periodic if and only if point (x, y) is the image of the origin after a finite number of reflections and lies on a horizontal edge. We now describe this set of parallel edges.

Highlighting all images of the edge that contains the origin reveals a tessellation of the plane by hexagons.

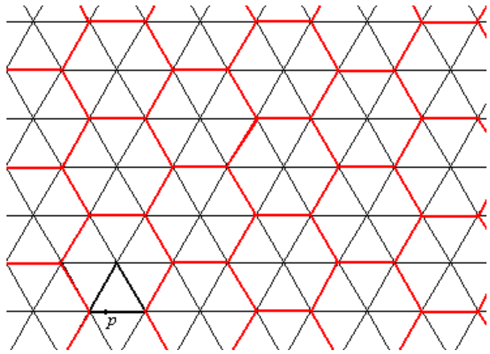


Figure 11: The hexagonal tessellation given by an equilateral tessellation.

We now find two linearly independent vectors which act as a basis for all vectors representing periodic orbits. The vectors $(-1, 2)$ and $(1, 1)$ form such a basis. Both are images of the origin and lie on horizontal edges and so represent periodic orbits. Any image of the origin that lies on a horizontal edge can be written as $a(-1, 2) + b(1, 1)$ for some integers a and b. Since $-1 \equiv 2$ and $1 \equiv 1 (\bmod 3)$, for $a, b \in \mathbb{Z}$ if $(x, y) = a(-1, 2) + b(1, 1)$ (i.e. is a periodic orbit) then $x \equiv y (\bmod 3)$.

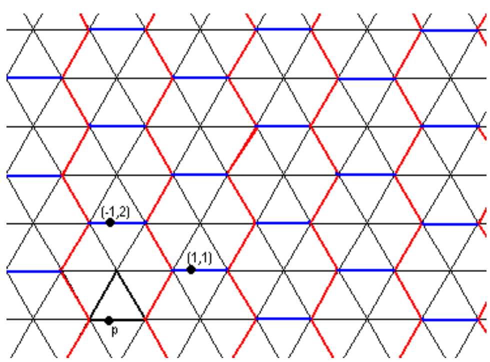


Figure 12: Periodic orbits initiate at P and terminate on a blue edge.

■

Given a vector representing a periodic orbit, we can determine the period by counting the number of lines the vector crosses. The vector of a period n orbit has a terminal point on the bounce circle of radius n, as is expected. The following theorem makes calculating the period very easy.

Theorem 6 *The period of a periodic orbit (x, y), $y > 0$, is given by*

$$Period(x,y) = \begin{cases} 2(x+y), & x \geq 0, y > 0 \\ 2y, & x < 0, y > -x \\ -2x, & x < 0, 0 < y \leq -x \end{cases} \tag{3}$$

Note that when $y < 0$, $Period(x, y) = Period(x, -y)$ by symmetry of the tessellation.

Proof. We derive the formula by considering each of the three cases in turn.

<u>*Case* 1:</u> Let $x \geq 0, y > 0$.

Let Γ be a periodic orbit from one horizontal to another. Cover Γ with parallelograms bounded by two horizontals and two right-leaning diagonals such that no two parallelograms intersect. A left-leaning diagonal cuts each parallelogram into two equilateral triangles.

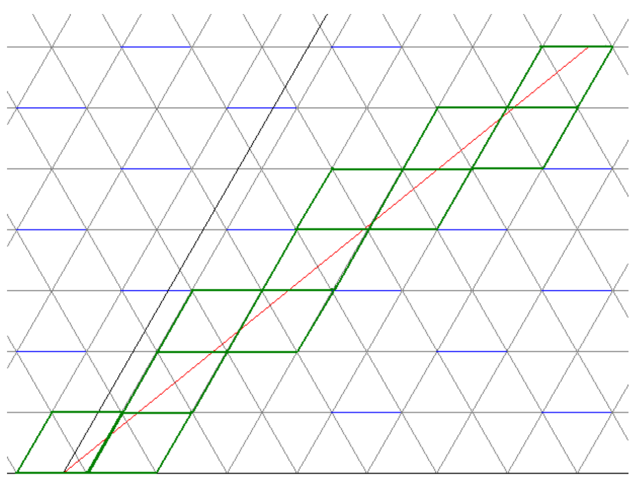


Figure 13: Parallelograms covering the vector $\Gamma = (4, 7)$.

For any parallelogram, Γ must pass through either the left or bottom side (exclusive) and through either the top or right side (exclusive), by the restriction of x and y. Thus, Γ must pass through the left-leaning diagonal contained in the parallelogram. Hence a left-leaning diagonal is crossed for each horizontal and right-leaning diagonal crossed.

Let h, r, and l be the number of horizontal lines, right-leaning diagonals, and left-leaning diagonals crossed. By inspection, $h = y$ and $r = x$, and we have just shown that $l = h + r$.

Therefore

$$Period(x, y) = h + r + l = y + x + (y + x) = 2(x + y).$$

<u>*Case* 2:</u> $x < 0, y > -x$.

We cover Γ with parallelograms as in case 1, except the parallelograms are bounded by two right-leaning diagonals and two left-leaning diagonals and split by a horizontal. Furthermore, we must cut one of the parallelograms in half, placing the top half on the original triangle and the bottom half such that its horizontal aligns with the edge containing the terminal point p'. Now one horizontal is crossed for each left- or right-leaning diagonal crossed.

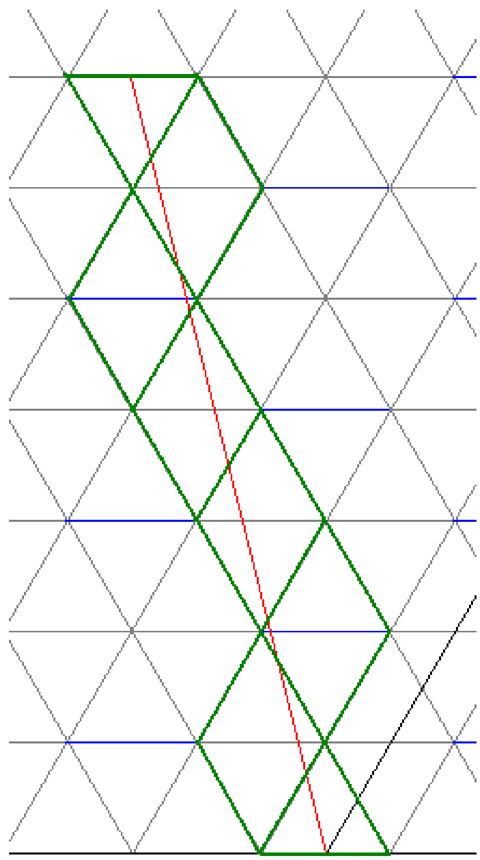


Figure 14: Parallelograms covering the vector $\Gamma = (-5, 7)$.

We now have $h = r + l$, or rather, $l = h - r$. Since $x < 0$, $r = -x$, so

$$Period(x, y) = h + r + l = y - x + (y + x) = 2y.$$

<u>*Case* 3:</u> $x < 0, 0 < y \leq -x$.

Again, we cover Γ by parallelograms, except the parallelograms are bounded by two left-leaning diagonals and two horizontals and split by a right-leaning diagonals.

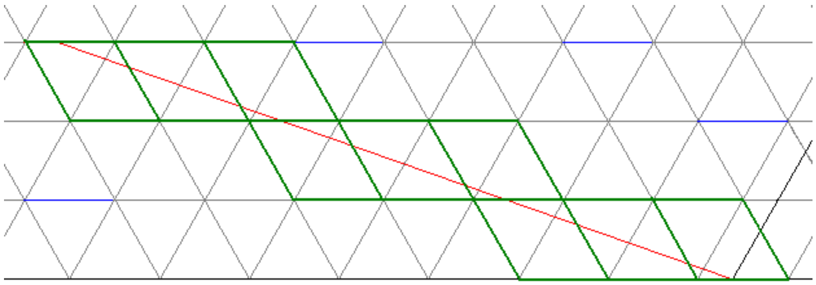


Figure 15: Parallelograms covering the vector $\Gamma = (-9, 3)$.

This makes $r = h + l$, or rather, $l = r - h$. Again, $r = -x$ since $x < 0$. Therefore

$$Period(x, y) = h + r + l = y - x + (-x - y) = -2x.$$

■

Dealing with three period formulas quickly complicates matters. Fortunately we may restrict our focus to a region requiring only one of the three.

Proposition 7 *For any orbit (not necessarily periodic) on the equilateral triangle, there are no more than three different bounce angles, with at least one between* $30°$ *and* $60°$*, inclusive.*

Proof. The tessellation consists of three sets of parallel lines, each intersecting the other two at $60°$. Orbit Γ could either run parallel to one of the three sets or cut through all three.

Γ cuts through all three lines when $\alpha < 60°$.

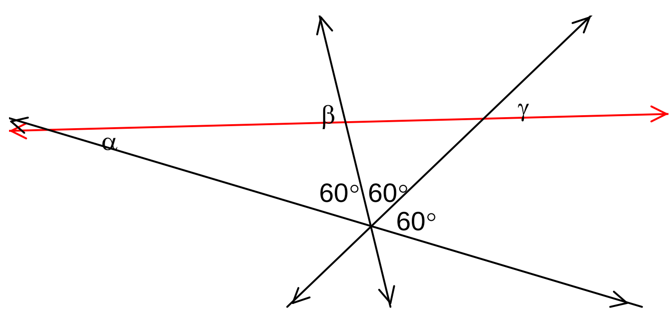


Figure 16: Γintersects the three lines at angles α, β and γ.

Trigonometry reveals that $\beta = 60° + \alpha$ and $\gamma = 60° - \alpha$, so either α or γ lies in the closed interval $[30°, 60°]$.

When $\alpha = 60°$, Γ runs parallel to one of the diagonals and $\alpha \in [30°, 60°]$. ■

Remark 8 *We can focus on periodic orbits with initial angle between* $30°$ *and* $60°$*, inclusive. In terms of the coordinates, this outlines the region* $x \geq 0, y \geq x$*.*

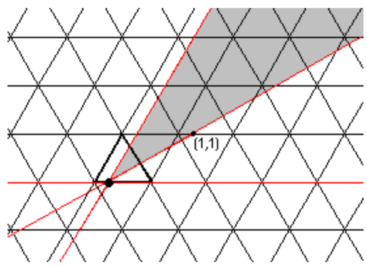


Figure 17: The region $0 \leq x \leq y$*.*

5 Counting Coordinate Pairs

We can now ask some interesting questions: Is there a periodic orbit of period $2n$ for any $n \in \mathbb{N}$? If so, how many periodic orbits are there of period $2n$ in all?

In addressing these questions, we make a simplifying assumption: If we repeat a period n orbit k times, we count the entire path as a period kn orbit. This is known as a k-fold duplication of a period n orbit, or a period kn orbit containing k duplicates. For example, if a period 4 orbit is repeated a second time, the whole orbit is counted as a period 8 orbit since it repeats itself after 8 bounces. This is akin to claiming that the function $\tan(x)$, a function of period π, has period 2π or even $k\pi$ for any natural number k. Stripping out these "orbits containing duplicates" is addressed later.

The question of existence is easy. Obviously there is no period 2 orbit since that requires the triangle to have a pair of parallel sides. Let $n \geq 2$ be a natural number. If n is odd, then $\left(\frac{n-3}{2}, \frac{n+3}{2}\right)$ forms a period $2n$ orbit. If n is even, then $\left(\frac{n}{2}, \frac{n}{2}\right)$ is a period $2n$ orbit.

To illustrate the second question, consider $n = 11$, that is, the period 22 orbits. Two vectors yield period 22 orbits: $(1, 10)$ and $(4, 7)$ (see Figure 18). How do we know these two orbits are not permutations of the same orbit? Calculating their lengths by equation 1 shows they are distinct. Alternately, one can calculate bounce angles by combining equation 2 and Proposition 7 to show the two orbits are distinct.

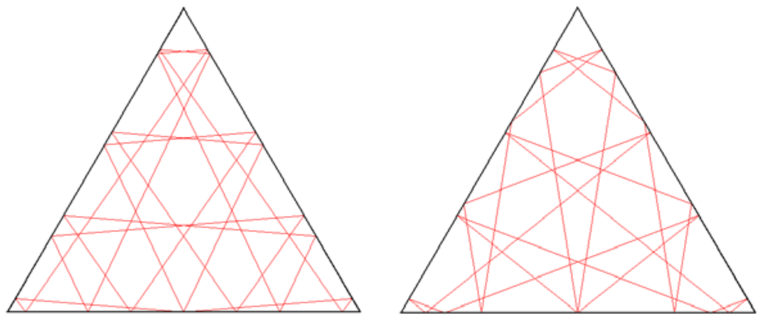


Figure 18: Two period 22 orbits: $(1, 10)$ (left) and $(4, 7)$ (right).

We now set about proving the following proposition:

Proposition 9 *Let $\lfloor x \rfloor$ denote the greatest integer function. Given $n \in \mathbb{N}$ such that $n > 1$, there exist $P(n) = \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n+2}{3} \rfloor$ period $2n$ orbits.*

This proposition is a special case of a more general combinatorial problem. Given a nonnegative integer n and natural number $m > 1$, how many ways can n be partitioned into two nonnegative numbers $a \leq b$ such that $a + b = n$ and $a \equiv b \pmod{m}$? Proposition 9 considers the case $m = 3$ and $n \geq 2$, as the pairs (a, b) are the vectors for period $2(a+b) = 2n$ orbits.

Definition 10 *Let m and n be integers such that $m \geq 2$ and $n \geq 0$. A partition of n modulo m is a pair of nonnegative integers $a \leq b$ such that $a + b = n$ and $a \equiv b \pmod{m}$. Let $P(n, m)$ denote the number of partitions of n modulo m.*

Theorem 11 *Let m, n, and r be integers such that $m \geq 2$, $n \geq 0$, and r is the least nonnegative residue of $\frac{n(m+1)}{2}$ modulo m. Then*

$$P(n, m) = \begin{cases} 0, & m \text{ even, } n \text{ odd} \\ \left\lfloor \frac{n}{m} \right\rfloor + 1, & m \text{ and } n \text{ even} \\ \left\lfloor \frac{1}{m} \left(\frac{n}{2} - r \right) \right\rfloor + 1, & m \text{ odd.} \end{cases} \tag{4}$$

Proof. Let $m, n \in \mathbb{Z}$ such that $m \geq 2$, $n \geq 0$. Suppose $a + b = n$ and $a \equiv b \pmod{m}$. Then $2a \equiv 2b \equiv n \pmod{m}$. We proceed by cases.

Case 1: m is even, n is odd.

Since $2a \equiv n \pmod{m}$ implies a contradiction, there are no partitions of n modulo m.

Case 2: m and n are even.

Now $2a \equiv 2b \equiv n \pmod{m}$ implies $a \equiv b \equiv \frac{n}{2} \pmod{\frac{m}{2}}$ by modular arithmetic. Any partition of n modulo m has the form $(\frac{n}{2} - \frac{mi}{2}, \frac{n}{2} + \frac{mi}{2})$ for $0 \leq i \leq \frac{n}{m}$. Therefore, there are $\lfloor \frac{n}{m} \rfloor + 1$ partitions of n modulo m.

Case 3: m is odd.

Now $2a \equiv 2b \equiv n \pmod{m}$ implies that $a \equiv b \equiv 2^{-1}n \equiv \frac{m+1}{2}n \pmod{m}$. Let $r \in \{0, 1, 2, \ldots, m-1\}$ such that $r \equiv \frac{m+1}{2}n \pmod{m}$. Then any partition of n modulo m has the form $(r + im, n - (r + im))$ for $0 \leq i \leq \frac{1}{m}\left(\frac{n}{2} - r\right)$. Therefore there are $\lfloor \frac{1}{m}\left(\frac{n}{2} - r\right) \rfloor + 1$ such partitions. ■

See Chart 1 in Appendix B for sample values for $P(n, m)$. This unsightly formula creates surprisingly simple recursive sequences, which in turn have elegant generating functions.

Corollary 12 *Given integer $m \geq 2$, the sequence $P_n = P(n, m)$ has the following recursion relations:*

1. *If m is odd, $P_n = P_{n-2m} + 1$.*
2. *If m is even, $P_{2n} = P_{2n-m} + 1$.*
3. *$P_n = P_{n-2} + \chi_0(n)$, where χ_0 is the characteristic function on the congruence class of 0 modulo m, $\chi_0(n) = \begin{cases} 1 & n \equiv 0 \mod m \\ 0 & \textit{otherwise.} \end{cases}$*

Proof. The first two relations can be derived from the formula for $P(n, m)$ by induction. We will only prove the third, as it is the most general.

Suppose $(a_1, b_1), (a_2, b_2), \ldots, (a_{P_n}, b_{P_n})$ are the partitions of n modulo m. Then $(a_1 + 1, b_1 + 1), (a_2 + 1, b_2 + 1), \ldots, (a_{P_n} + 1, b_{P_n} + 1)$ are each partitions of $n + 2$ modulo m. Additionally, if $n \equiv 0 \pmod{m}$ then $(0, n)$ also partitions n modulo m.

To be sure that all partitions of n modulo m have been counted, suppose (a, b) is a partition of n modulo m, where $a > 0$. Then $(a - 1, b - 1)$ is a partition of $n - 2$ modulo m, so $a - 1 = a_i$ and $b - 1 = b_i$ for some $i \in \{1, 2, \ldots, P_{n-2}\}$. Thus $P_n = P_{n-2} + \chi_0(n)$. ■

Remark 13 *Every sequence P_n has initial terms $P_0 = 1$ (the partition (0,0)) and $P_1 = 0$ ($0 \neq 1$ modulo $m > 1$).*

Corollary 14 *Given integer $m > 1$, the sequence $P_n = P(n, m)$ has the generating function $\frac{1}{(1-x^2)(1-x^m)}$.*

Proof. Let $m \geq 2$ be a natural number. Define A_n such that

$$\sum_{n=0}^{\infty} A_n x^n = \frac{1}{(1-x^2)(1-x^m)}. \tag{5}$$

Then $(1-x^2)\sum_{n=0}^{\infty} A_n x^n = \frac{1}{(1-x^m)}$. Distributing on the left and replacing $\frac{1}{(1-x^m)}$ with its generating function $\sum_{n=0}^{\infty} \chi_0(n)x^n$, we are left with

$$\begin{aligned}
\sum_{n=0}^{\infty} \chi_0(n)x^n &= \sum_{n=0}^{\infty} A_n x^n - \sum_{n=0}^{\infty} A_n x^{n+2} && (6)\\
&= A_0 + A_1 x + \sum_{n=2}^{\infty} A_n x^n - \sum_{n=2}^{\infty} A_{n-2} x^n && (7)\\
&= A_0 + A_1 x + \sum_{n=2}^{\infty} (A_n - A_{n-2}) x^n. && (8)
\end{aligned}$$

Therefore, $A_0 = \chi_0(0) = 1$, $A_1 = \chi_0(1) = 0$, and $A_n - A_{n-2} = \chi_0(n)$ for all $n \geq 2$. By Corollary 12 and Remark 13 with $A_n = P_n$ for all $n \geq 0$, $\frac{1}{(1-x^2)(1-x^m)}$ is the generating function for P_n. ■

The combinatorics-savvy reader will recognize the generating function $\frac{1}{(1-x^2)(1-x^m)}$ as enumerating the number of partitions of n using only 2's and m's as the summands. The bijection between the sets of partitions is discussed in the Appendix.

We are now ready to prove Proposition 9.

Proof. Let $P(n) = \lfloor \frac{n+2}{2} \rfloor - \lfloor \frac{n+2}{3} \rfloor$. We show that $P(n)$ has the same recursion relation and initial conditions as in Corollary 12 and Remark 13 showing $P(n) = P(n, 3)$ for all integers $n \geq 0$.

First confirm by calculation that $P(0) = 1$ and $P(1) = 0$.

Let $n \geq 2$

$$P(n-2) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor \tag{9}$$
$$= \left\lfloor \frac{n+2}{2} \right\rfloor - 1 - \left\lfloor \frac{n}{3} \right\rfloor \tag{10}$$
$$= \left\lfloor \frac{n+2}{2} \right\rfloor - \left(\left\lfloor \frac{n}{3} \right\rfloor + 1 \right). \tag{11}$$

By examination, we see that $\lfloor \frac{n}{3} \rfloor + 1 = \lfloor \frac{n+2}{3} \rfloor + \chi_0(n)$.

$$\left\lfloor \frac{n}{3} \right\rfloor + 1 = \begin{cases} \frac{n}{3} + 1, & n \equiv 0 (\operatorname{mod} 3) \\ \frac{n-1}{3} + 1, & n \equiv 1 (\operatorname{mod} 3) \\ \frac{n-2}{3} + 1, & n \equiv 2 (\operatorname{mod} 3) \end{cases}$$

$$\left\lfloor \frac{n+2}{3} \right\rfloor + \chi_0(n) = \begin{cases} \frac{(n+2)-2}{3} + 1, & n \equiv 0 (\operatorname{mod} 3) \\ \frac{(n+2)}{3} + 0, & n \equiv 1 (\operatorname{mod} 3) \\ \frac{(n+2)-1}{3} + 0, & n \equiv 2 (\operatorname{mod} 3) \end{cases}$$

Therefore, $P(n-2) = P(n) + \chi_0(n)$, so the recursion relation holds. ■

Note that the terminal points of all vectors of period $2n$ orbits lie between the same two left-leaning diagonals.

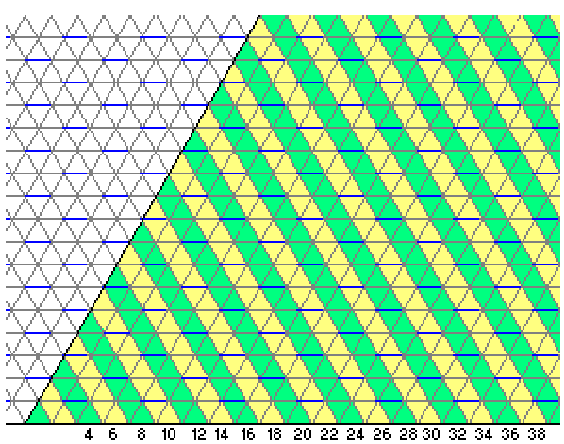


Figure 19: Orbits in a colored band have the same period.

6 Stripping out Orbits Containing Duplicates

We now reevaluate the assumption we made when counting orbits of a given period.

Definition 15 *Given periodic orbit $\Gamma = (x, y)$, let $d \in \mathbb{N}$ be the largest value such that $x/d \equiv y/d (\bmod 3)$. If $d = 1$, then Γ is a* <u>*duplicate-free orbit*</u>*; otherwise, Γ is an* <u>*orbit containing d duplicates*</u>*.*

Finding such a d is cumbersome when x and y are very large. Fortunately the coordinates' common factors determine whether an orbit is duplicate-free.

Theorem 16 *An orbit with vector (x, y) is duplicate-free if and only if one of the following is true:*

1. *x and y are relatively prime.*
2. *If $(x, y) = (3a, 3b)$ for $a, b \in \mathbb{Z}$, then $a \neq b(mod 3)$ and a and b are relatively prime.*

Proof. *Part 1:* Suppose (x, y) represents a periodic orbit such that integers x and y are relatively prime. Then the only integer that divides both x and y is 1, and (x, y) must be duplicate-free by Definition 15.

Conversely, suppose (x, y) is duplicate-free and $3 \nmid x$ and $3 \nmid y$. Let $d = \gcd(x, y)$. Then $3 \nmid k$ as well.

$$x \equiv y (\bmod 3) \tag{12}$$

$$dm \equiv dn (\bmod 3), \text{ for some } m, n \in \mathbb{Z} \tag{13}$$

$$m \equiv n (\bmod 3) \tag{14}$$

Then $(m, n) = (\frac{x}{d}, \frac{y}{d})$ is periodic, but since (x, y) is duplicate-free, $d = 1$. Therefore x and y are relatively prime.

Part 2: Suppose $(x, y) = (3a, 3b)$ represents a periodic orbit for integers a and b such that $a \neq b$ (mod 3) and a and b are relatively prime. Then $(x, y) = d\left(\frac{x}{d}, \frac{y}{d}\right)$ for $d \in \mathbb{N}$ implies $d = 1$ or $d = 3$. If $d = 3$, then $\left(\frac{x}{d}, \frac{y}{d}\right) = (a, b)$, which is not a periodic orbit since $a \neq b$ (mod 3). $d = 1$ implies that (x, y) is duplicate-free. Therefore, (x, y) is duplicate-free.

Now suppose $(x, y) = (3a, 3b)$ is duplicate-free. Then (a, b) is not periodic, so $a \not\equiv b \pmod 3$. Secondly, suppose $d \mid a$ and $d \mid b$. Since $\frac{3a}{d} \equiv \frac{3b}{d} \pmod 3$, the orbit $(\frac{x}{d}, \frac{y}{d})$ is periodic. But (x, y) is duplicate-free, however, so $d = 1$. ■

We first answer whether a duplicate-free period $2n$ orbit exists for all n, then conclude this section by giving an exact count.

Theorem 17 *Let $n \geq 2$ be a natural number. Then there exists a duplicate-free period $2n$ orbit if and only if $n \neq 4, 6,$ or 10.*

Proof. Recall $\gcd(x, y) \mid (y - x)$.

Case 1: n is odd.

The duplicate-free orbit is $\left(\frac{n-3}{2}, \frac{n+3}{2}\right)$. A quick check confirms that $\frac{n-3}{2} \equiv \frac{n+3}{2} \pmod 3$. Now $\frac{n+3}{2} - \frac{n-3}{2} = 3$, so if $\frac{n-3}{2}$ is not a multiple of 3 then $\gcd(\frac{n-3}{2}, \frac{n+3}{2}) = 1$.

If $n = 3m$ for some (odd) integer m then $(\frac{n-3}{2}, \frac{n+3}{2}) = \left(3\left(\frac{m-1}{2}\right), 3\left(\frac{m+1}{2}\right)\right)$. Since $a = \frac{m-1}{2}$ and $b = \frac{m+1}{2}$ are consecutive integers, $\gcd(a, b) = 1$ and $a \not\equiv b \pmod 3$. Therefore $\left(\frac{n-3}{2}, \frac{n+3}{2}\right)$ is duplicate-free.

Case 2: $n \equiv 0 \pmod 4$

If $n \neq 4$, the duplicate-free orbit is $\left(\frac{n}{2} - 3, \frac{n}{2} + 3\right)$. This is obviously a periodic orbit. $\frac{n}{2} + 3 - (\frac{n}{2} - 3) = 6$, so if n is not a multiple of 3 then the two coordinates are relatively prime. Note that even though 2 divides the difference, 2 cannot divide either coordinate since both are odd.

If n is a multiple of 3, then $n = 12m$ for some $m \in \mathbb{Z}$, $\frac{n}{2} - 3 = 3(2m - 1)$, and $\frac{n}{2} + 3 = 3(2m + 1)$. Since $a = 2m - 1$ and $b = 2m + 1$ are consecutive odd integers, $\gcd(a, b) = 1$ and $a \not\equiv b \pmod 3$.

If $n = 4$, then $x = \frac{n}{2} - 3 = -1$, which is outside the bounds we set in Remark 8. Upon inspection, we find (2,2) is the only mod-3 partition of 4, which is twice the orbit (1,1). Thus there is no periodic orbit of period $2 \times 4 = 8$.

Case 3: $n \equiv 2 \pmod 4$

If $n \geq 14$, the duplicate-free orbit is $\left(\frac{n}{2} - 6, \frac{n}{2} + 6\right)$. $\frac{n}{2} + 6 - (\frac{n}{2} - 6) = 12$. Since both coordinates are odd, if $\frac{n}{2} - 6$ is not an multiple of 3, then $\gcd(\frac{n}{2} - 6, \frac{n}{2} + 6) = 1$.

If $\frac{n}{2}-6$ is a multiple of three, then $n = 6m$ for some (odd) $m \in \mathbb{Z}$. Then $\frac{n}{2}-6 = 3(m-2)$ and $\frac{n}{2}+6 = 3(m+2)$. Since $(m+2)-(m-2) = 4$ and m is odd, $\gcd(m-2, m+2) = 1$ and $m-2 \neq m+2 \pmod 3$.

If $n = 2, 6$, or 10, $x = \frac{n}{2} - 6$, lies outside the bound $x \geq 0$. By listing partitions modulo 3 for $n = 6$ and $n = 10$ as in case 2, we see that neither has duplicate-free periodic orbits. The case $n = 2$ is special, however, as it does have the duplicate-free periodic orbit (1,1). ■

Enumerating duplicate-free period $2n$ orbits is most easily done by counting periodic orbits containing duplicates and subtracting this number from $P(n)$. We begin with a lemma.

Lemma 18 *A period $2n$ orbit contains a d-fold duplicate only if $d \mid n$. Each duplicate has period $\frac{2n}{d}$.*

Proof. If period $2n$ orbit (x, y) contains d duplicates, then $\left(\frac{x}{d}, \frac{y}{d}\right)$ is a periodic orbit (though not necessarily duplicate-free). Since $d \mid x$ and $d \mid y$, $d \mid (x+y) = n$. Furthermore,

$$\mathit{Period}\left(\frac{x}{d}, \frac{y}{d}\right) = 2\left(\frac{x}{d} + \frac{y}{d}\right) = 2\frac{(x+y)}{d} = \frac{2n}{d}. \tag{15}$$

■

From Proposition 9, there are $P\left(\frac{n}{d}\right)$ period $2n$ orbits containing d duplicates. By examining n's prime factors we can account for all orbits containing duplicates.

Let $n = p_1^{r_1} p_2^{r_2} \ldots p_m^{r_m}$, where $p_1, p_2, \ldots, p_m$ are prime. Then there are $P\left(\frac{n}{p_1}\right)$ p_1-fold duplicates, $P\left(\frac{n}{p_2}\right)$ p_2-fold duplicates, down to $P\left(\frac{n}{p_m}\right)$ p_m-fold duplicates. However $(p_1 p_2)$-fold duplicates, for example, have been counted twice, so we must subtract $\sum P\left(\frac{n}{p_i p_j}\right)$ for all pairs $i < j$. Now we have subtracted *too many* and must add $(p_i p_j p_k)$-fold duplicates $i < j < k$ back in. Continue this process of alternately adding and subtracting as per the Principle of Inclusion-Exclusion until reaching $p_1 p_2 \cdots p_m$-fold duplicates. The final tally of the $2^m - 1$ terms is the total number of period $2n$ orbits containing duplicates. Let

$$D(n) = P(n) - \sum_{d \mid n} \mu(d) P\left(\frac{n}{d}\right),$$

where μ is the Möbius function given by

$$\mu(d) = \begin{cases} 1, & d = 1 \\ (-1)^r, & d = p_1 p_2 \cdots p_r \text{ with } p_i\text{'s distinct primes} \\ 0, & \text{otherwise.} \end{cases}$$

We have proved:

Theorem 19 *There are $D(n)$ period $2n$ orbits containing duplicates.*

Most importantly, we can now count the number of period $2n$ duplicate-free orbits.

Corollary 20 *There are $\sum_{d|n} \mu(d) P\left(\frac{n}{d}\right)$ duplicate-free period $2n$ orbits.*

Proof. Since every orbit is either duplicate-free or duplicate, there are

$$F(n) = P(n) - D(n) \tag{16}$$

duplicate-free period $2n$ orbits. Thus,

$$F(n) = P(n) - \left(P(n) - \sum_{d|n} -\mu(d) P\left(\frac{n}{d}\right) \right) = \sum_{d|n} \mu(d) P\left(\frac{n}{d}\right). \tag{17}$$

■

Example 21 *Let $n = 50 = 2 \cdot 5^2$. There are*

$$D(n) = P\left(\frac{50}{2}\right) + P\left(\frac{50}{5}\right) - P\left(\frac{50}{2 \cdot 5}\right) = P(25) + P(10) - P(5) = 4 + 2 - 1 = 5 \tag{18}$$

orbits containing duplicates. Listing all $P(50) = 9$ orbits, mark those that contain 2-, 5-, and 10-fold duplicates.

(x, y)	*2-fold*	*5-fold*	*10-fold*	*Duplicate-free*
$(1, 49)$				✓
$(4, 46)$	✓			
$(7, 43)$				✓
$(10, 40)$	✓	✓	✓	
$(13, 37)$				✓
$(16, 34)$	✓			
$(19, 31)$				✓
$(22, 28)$	✓			
$(25, 25)$		✓		

The table's columns illustrate there are 4 orbits that contain 2-fold duplicates, 2 orbits with 5-fold duplicates, and 1 with both 2- and 5-fold duplicates resulting in a 10-fold duplicate.

Example 22 *Let* $n = 44100 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$. *Then there are*

$$\begin{aligned}
&P\left(\frac{44100}{2}\right) + P\left(\frac{44100}{3}\right) + P\left(\frac{44100}{5}\right) + P\left(\frac{44100}{7}\right) \\
&-\left(P\left(\frac{44100}{2\cdot 3}\right) + P\left(\frac{44100}{2\cdot 5}\right) + P\left(\frac{44100}{2\cdot 7}\right) + P\left(\frac{44100}{3\cdot 5}\right) + P\left(\frac{44100}{3\cdot 7}\right) + P\left(\frac{44100}{5\cdot 7}\right)\right) \\
&+\left(P\left(\frac{44100}{2\cdot 3\cdot 5}\right) + P\left(\frac{44100}{2\cdot 3\cdot 7}\right) + P\left(\frac{44100}{2\cdot 5\cdot 7}\right) + P\left(\frac{44100}{3\cdot 5\cdot 7}\right)\right) \\
&-P\left(\frac{44100}{2\cdot 3\cdot 5\cdot 7}\right) \\
&= 3676 + 2451 + 1471 + 1051 - 1226 - 736 - 526 - 491 - 351 - 211 \\
&\qquad + 246 + 176 + 106 + 71 - 36 \\
&= 5671
\end{aligned}$$

period 88200 orbits containing duplicates, and

$$P(44100) - 5671 = 7351 - 5671 = 1680$$

duplicate-free orbits.

This formula has a few interesting special cases. For example,

Corollary 23 *There are no period* $2p$ *orbits containing duplicates if and only if* p *is prime.*

Proof. This follows directly from $P\left(\frac{p}{p}\right) = P(1) = 0$ and the fact that 1 is the only value of n such that $P(n) = 0$. ■

Chart 2 and Graph 1 in Appendix B show sample values for $P(n), D(n)$, and $F(n)$.

7 Conclusion

We have shown in Theorem 17 that the equilateral triangle has at least one duplicate-free period $2n$ orbit for every positive integer $n \neq 1, 4, 6$, or 10, and more precisely in Corollary 20 that there are $\sum_{d|n} \mu(d) P\left(\frac{n}{d}\right)$ duplicate-free period $2n$ orbits. Any function to describe the number of period $2n$ orbits containing duplicates (which could be used to find the number of duplicate-free orbits) that does not rely on divisors of n, that is one more resembling $P(n)$, would need to have roots at all prime numbers and only at prime numbers. The existence of such a function is not known.

Open questions remain. For example, a periodic orbit resembling $(0, 3)$, along with the degenerate period 3 case, exists on any acute triangle. Likewise the period 4 orbit $(1, 1)$ appears on any isosceles triangle. The reflections that produce the period 10 orbit in Figure 4 also produce a period 10 orbit on any acute isosceles triangle. This indicates that many of the results above could find generalizations that apply to all acute isosceles triangles.

Another open question lies more in the realm of analytic number theory. Define $\mathcal{P}(n) = \sum_{i=1}^{n} P(n)$ and $\mathcal{F}(n) = \sum_{i=1}^{n} F(n)$. Upon inspection of their graphs, both functions appear to be approximately quadratic. A more interesting question, perhaps, considers the end behavior of the function $\mathcal{Q}(n) = \frac{\mathcal{F}(n)}{\mathcal{P}(n)}$. Obviously $\mathcal{Q}(n)$ cannot exceed 1, but does it approach some limit $L < 1$. The expression $\lim_{n\to\infty} \mathcal{Q}(n)$ gives the proportion of all periodic orbits (of any period) which contain no duplicates.

8 Appendices

8.1 Appendix A: A Bijection

Recall Definition 10 and the remarks following Corollary 14. We describe a bijection between the partitions modulo m of nonnegative integer n and the partition of n into summands 2 and m, here denoted $(2, m)$-partitions.

Let (a, b) be a partition of n modulo m. Without loss of generality, let $a \leq b$. Then $a \times 2 + \frac{(b-a)}{m} \times m$ is a $(2, m)$-partition, since $m \mid (b - a)$.

For the inverse, let $a \times 2 + b \times m$ be a $(2, m)$-partition of n. Then $(a, a + bm)$ is a partition of n modulo m.

The bijection is easiest seen with a Ferrers diagram. Consider the case $n = 8$ and $m = 3$. The partitions modulo m are (1,7) and (4,4), whereas the $(2, m)$-partitions are 2+3+3 and 2+2+2+2.

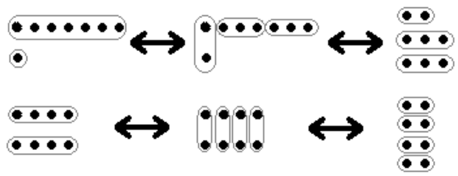


Figure 30: Ferrers diagram for $n = 8$ and $m = 3$.

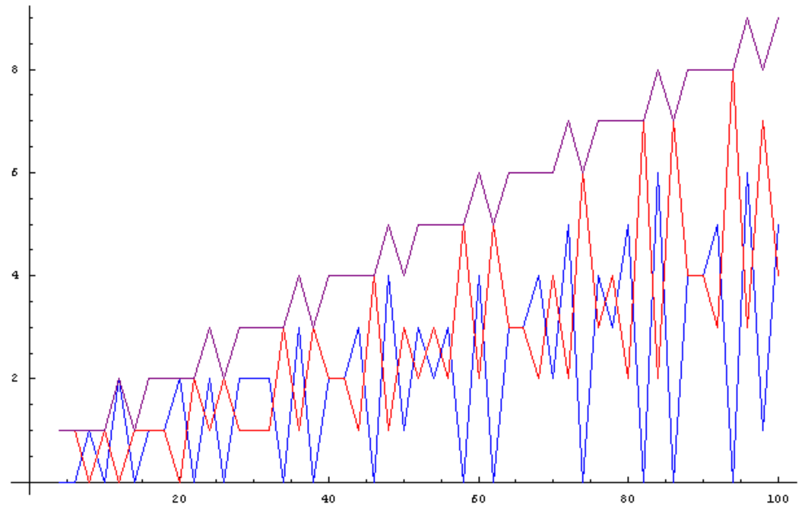
8.2 Appendix B: Charts and Graphs

$_{n}\ ^{m}$	2	3	4	5	6	7	8	$_{n}\ ^{m}$	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	20	11	4	6	3	4	2	3
1	0	0	0	0	0	0	0	21	0	4	0	2	0	2	0
2	2	1	1	1	1	1	1	22	12	4	6	3	4	2	3
3	0	1	0	0	0	0	0	23	0	4	0	2	0	2	0
4	3	1	2	1	1	1	1	24	13	5	7	3	5	2	4
5	0	1	0	1	0	0	0	25	0	4	0	3	0	2	0
6	4	2	2	1	2	1	1	26	14	5	7	3	5	2	4
7	0	1	0	1	0	1	0	27	0	5	0	3	0	2	0
8	5	2	3	1	2	1	2	28	15	5	8	3	5	3	4
9	0	2	0	1	0	1	0	29	0	5	0	3	0	2	0
10	6	2	3	2	2	1	2	30	16	6	8	4	6	3	4
11	0	2	0	1	0	1	0	31	0	5	0	3	0	2	0
12	7	3	4	2	3	1	2	32	17	6	9	4	6	3	5
13	0	2	0	1	0	1	0	33	0	6	0	3	0	2	0
14	8	3	4	2	3	2	2	34	18	6	9	4	6	3	5
15	0	3	0	2	0	1	0	35	0	6	0	4	0	3	0
16	9	3	5	2	3	2	3	36	19	7	10	4	7	3	5
17	0	3	0	2	0	1	0	37	0	6	0	4	0	3	0
18	10	4	5	2	4	2	3	38	20	7	10	4	7	3	5
19	0	3	0	2	0	1	0	39	0	7	0	4	0	3	0

Chart 1: Sample values for $P(n, m)$

n	2n	P(n)	F(n)	D(n)	n	2n	P(n)	F(n)	D(n)
2	4	1	1	0	21	42	4	2	2
3	6	1	1	0	22	44	4	1	3
4	8	1	0	1	23	46	4	4	0
5	10	1	1	0	24	48	5	1	4
6	12	2	0	2	25	50	4	3	1
7	14	1	1	0	26	52	5	2	3
8	16	2	1	1	27	54	5	3	2
9	18	2	1	1	28	56	5	2	3
10	20	2	0	2	29	58	5	5	0
11	22	2	2	0	30	60	6	2	4
12	24	3	1	2	31	62	5	5	0
13	26	2	2	0	32	64	6	3	3
14	28	3	1	2	33	66	6	3	3
15	30	3	1	2	34	68	6	2	4
16	32	3	1	2	35	70	6	4	2
17	34	3	3	0	36	72	7	2	5
18	36	4	1	3	37	74	6	6	0
19	38	3	3	0	38	76	7	3	4
20	40	4	2	2	39	78	7	4	3

Chart 2: Sample values for the total number of orbits ($P(n)$), duplicate-free orbits ($F(n)$), and orbits containing duplicates ($D(n)$).



Graph 1: Plot of Chart 2. $P(n)$ is shown in purple, $F(n)$ in red, and $D(n)$ in blue.

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