Partitioning Integers into k Non-Negative Addends Congruent Modulo m

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Abstract: For any nonnegative integer n, we partition n into k addends, each of which are congruent to one another modulo m, where m is an natural number. We find that the problem is equivalent to partition n using only $k, m, 2m, \ldots, (k-1)m$ as parts, whose generating function is already known. We develop alternate generating functions, recurrence relations, and some explicit formulas for the special cases $k = 2$, $k = m$, and $k = lm$ for some integer $l > 1$.

1 Introduction

Let a <u>k-part modulo m partition</u> be an unordered integer partition of n into k nonnegative integers congruent modulo m, that is, the number of multisets $\{a_1, a_2, \ldots, a_k\}$ such that $\sum a_i = n$ and $a_i \equiv a_j \mod m$ for all $1 \le i \le j \le k$. Let $P_{k,m}(n)$ denote the number of such partitions.

This problem, while on the surface new, is rather equivalent to the known problem of partitioning n using only $k, m, 2m, \ldots, (k-1)m$ as parts. We derive a bijection between these two types of integer partitions in section 2, as well as discuss the generating function that is known. From there we explore special cases for the values of k and m. In section 3, we find an explicit formula for $k = 2$ and $m > 2$, as well as a trio of recurrence relations. Section 4 explores when $k = m$, giving a recurrence relation and explicit formulas for $k = 1, 2, 3, 4$, and 5. In section 5 we extend some results of section 4 as we consider the case $k = lm$ for some integer $l > 1$.

2 Equivalence to a Known Problem

Theorem 1 *The number of k-part modulo m partitions of n is equal to the number of partitions of n using only $k, m, 2m, \ldots, (k-1)m$ as parts.*

Proof. We create a bijection, B, from k-part modulo m partitions to partitions of n using only $k, m, 2m, \ldots, (k-1)m$ as parts.

Let $a_1 + a_2 + \cdots + a_k$ be a partition of n such that $a_1 \equiv a_2 \equiv \cdots \equiv a_k$ mod m and $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$. Let $b_1, b_2, \ldots, b_k$ be nonnegative integers such that $b_1 m + b_2 2m + \cdots b_{k-1}(k-1)m + b_k k = n$.

Define $f : (a_1, a_2, \ldots, a_k) \mapsto (b_1, b_2, \ldots, b_k)$ as follows. Let $b_i = \frac{a_i - a_{i+1}}{m}$ for $i = 1, 2, 3, \ldots, k-1$. Let $b_k = a_k$.

Suppose $B(a_1, a_2, \ldots, a_k) = B(A_1, A_2, \ldots, A_k)$. Let $B(a_1, a_2, \ldots, a_k) = (b_1, b_2, \ldots, b_k)$ and $B(A_1, A_2, \ldots, A_k) = (B_1, B_2, \ldots, B_k)$. Then $b_i = B_i$ for all $1 \leq i \leq k$, so $\frac{a_i - a_{i+1}}{m} = \frac{A_i - A_{i+1}}{m}$ for all $1 \leq i \leq k-1$ and $a_k = A_k$. Thus we get the following set of equivalences:

$$
\begin{aligned}
a_1 - a_2 &= A_1 - A_2 \\
a_2 - a_3 &= A_2 - A_3 \\
&\vdots \\
a_{k-1} - a_k &= A_{k-1} - A_k
\end{aligned}
$$

Since $a_k = A_k$, then $a_{k-1} = A_{k-1}$. Continuing to substitute reveals that $a_i = A_i$ for all i. Thus f is one-to-one.

Define $f^{-1} : (b_1, b_2, \ldots, b_k) \mapsto (a_1, a_2, \ldots, a_k)$ by letting $a_i = b_k + \sum\limits_{j=i}^{k-1} m b_j$ for $i = 1, 2, 3, \ldots k-1$ and $a_k = b_k$. We show that $f(f^{-1}(b_1, b_2, \ldots, b_k)) = (b_1, b_2, \ldots, b_k)$.

$$
f(f^{-1}(b_1, b_2, \ldots, b_k)) = f\left(b_k + \sum_{j=1}^{k-1} m b_j, b_k + \sum_{j=2}^{k-1} m b_j, \ldots, b_k + \sum_{j=k-1}^{k-1} m b_j, b_k\right) \tag{1}
$$

The i^{th} term, for $1 \leq i \leq k-1$ is

$$
\frac{b_k + \sum\limits_{j=i}^{k-1} m b_j - \left(b_k + \sum\limits_{j=i+1}^{k-1} m b_j\right)}{m} = \frac{m b_i}{m} = b_i \tag{2}
$$

Thus f is invertible. Therefore f is a bijection. ■

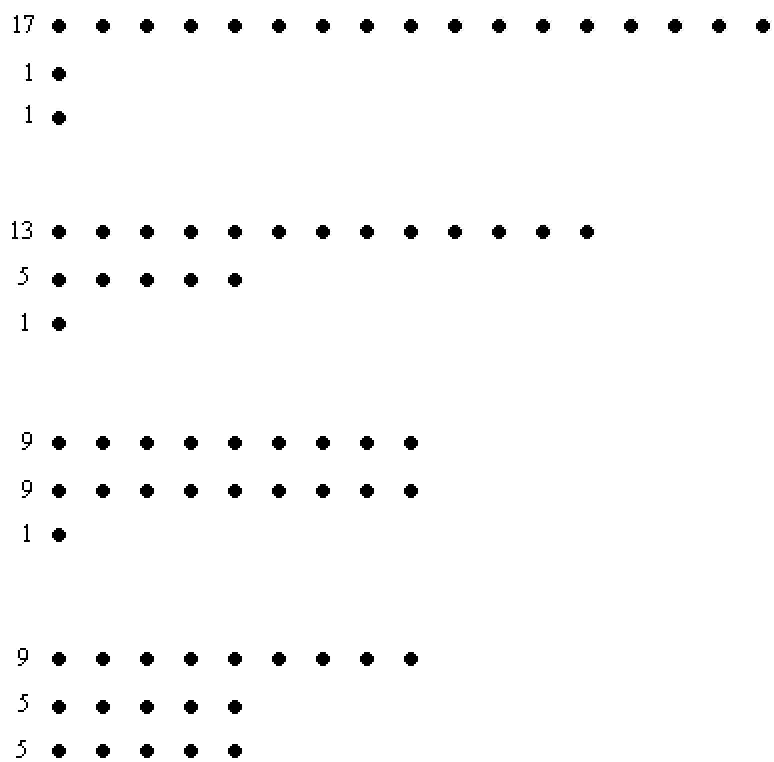
Note that if $k = m$, the partition into $k, m, 2m, \ldots,$ and $(k-1)m$ treats k and m as two distinct numbers with the same value. This special case is explored more in Section 4.

Example 2 *Let* $k = 3, m = 4$ *and* $n = 19$. *There are four such partitions.*

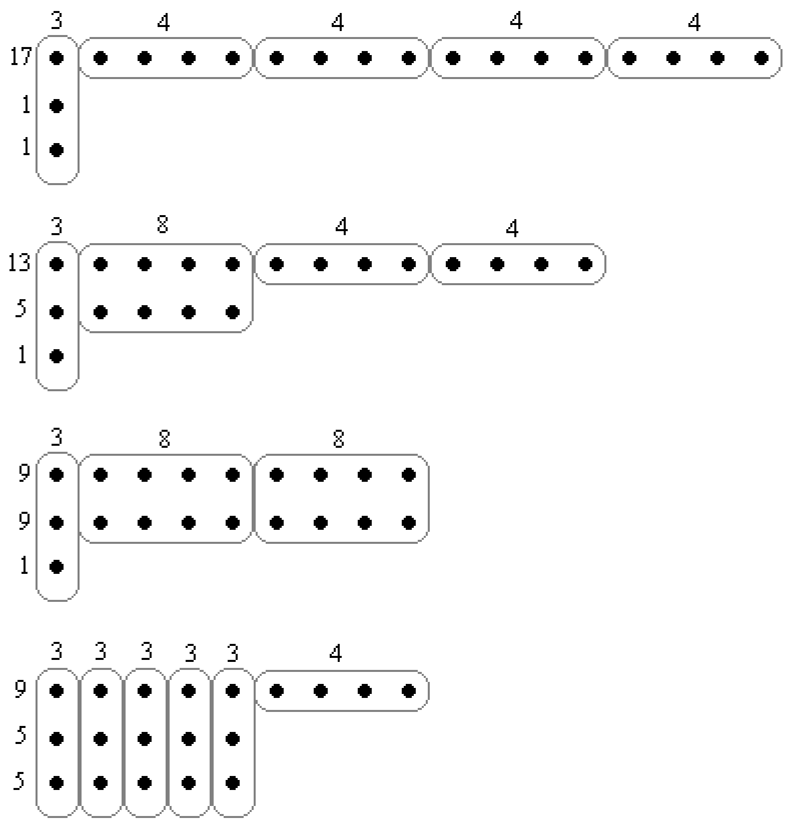
$a_1 + a_2 + a_3$		$b_1 \times 4 + b_2 \times 8 + b_3 \times k$	*Expanded and Reordered*
$17 + 1 + 1$	$\leftrightarrow$	$4 \times 4 + 0 \times 8 + 1 \times 3$	$= 3 + 4 + 4 + 4 + 4$
$13 + 5 + 1$	$\leftrightarrow$	$2 \times 4 + 1 \times 8 + 1 \times 3$	$= 3 + 4 + 4 + 8$
$9 + 9 + 1$	$\leftrightarrow$	$0 \times 4 + 2 \times 8 + 1 \times 3$	$= 3 + 8 + 8$
$9 + 5 + 5$	$\leftrightarrow$	$1 \times 4 + 0 \times 8 + 5 \times 3$	$= 3 + 3 + 3 + 3 + 3 + 4$

The algebra of the bijection may be a bit obfuscative, but it is to make the Ferrers diagram that much more intuitive.

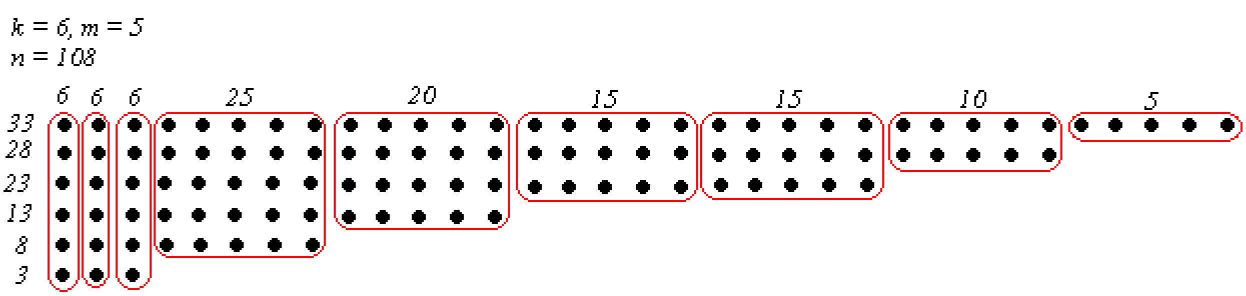
Consider the Ferrers diagram of the partitions $17+1+1, 13+5+1, 9+9+1,$ *and* $9 + 5 + 5$.



We can now regroup the dots to highlight the corresponding partitions $3 + 4 + 4 + 4 + 4, 3 + 8 + 4 + 4, 3 + 8 + 8,$ *and* $3 + 3 + 3 + 3 + 4$.



Because the rows are congruent to one another modulo m, each row is a multiple of m greater than the row below it. This allows us to draw boxes which contain a multiple of m dots. To illustrate this regrouping principle, consider the Ferrers diagram below showing the correspondence between 33+28+23+13+8+3 and 6+6+6+25+20+15+15+10+5 ($k = 6, m = 5$, and $n = 108$).



Note that this bijection holds for $n = 0$ if we count the null partition as a valid partition of n using $k, m, 2m, \ldots, (k-1)m$ as parts. The null partition is a partition using none of the available parts, giving a sum of zero. Obviously this is the only partition of zero into nonzero parts and only zero can be partitioned by the null partition. We count the null partition as a valid partition into given parts for the remainder of this article.

Corollary 3 *$P_{k,m}(n)$ has the following generating function:*

$$f(x) = \frac{1}{(1-x^k)\prod_{i=1}^{k-1}(1-x^{im})} \tag{3}$$

Proof. This is the generating function for the number of partitions of n using only $k, m, 2m, \ldots, (k-1)m$ as parts. ■

3 Special Case: $k = 2$

We first examine the special case when $k = 2$, effectively splitting n into two parts congruent modulo m. This simplifies the generating function to $\frac{1}{(1-x^2)(1-x^m)}$.

Theorem 4 *Let m, n, and r be integers such that $m \geq 2$, $n \geq 0$, and r is the least nonnegative residue of $\frac{n(m+1)}{2}$ modulo m. Then*

$$P_{2,m}(n) = \begin{cases} 0, & m \text{ even, } n \text{ odd} \\ \lfloor \frac{n}{m} \rfloor + 1, & m \text{ and } n \text{ even} \\ \lfloor \frac{1}{m}\left(\frac{n}{2} - r\right) \rfloor + 1, & m \text{ odd.} \end{cases} \tag{4}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Proof. Let $m, n \in \mathbb{Z}$ such that $m \geq 2$, $n \geq 0$. Suppose $a + b = n$ and $a \equiv b \pmod{m}$. Then $2a \equiv 2b \equiv n \pmod{m}$. We proceed by cases.
Case 1: m is even, n is odd.

Since $2a \equiv n \pmod{m}$ implies a contradiction, there are no partitions of n modulo m.
Case 2: m and n are even.

Now $2a \equiv 2b \equiv n \pmod{m}$ implies $a \equiv b \equiv \frac{n}{2} \pmod{\frac{m}{2}}$ by modular arithmetic. Any partition of n modulo m has the form $\left(\frac{n}{2} - \frac{mi}{2}\right) + \left(\frac{n}{2} + \frac{mi}{2}\right)$ for $0 \leq i \leq \frac{n}{m}$. Therefore, there are $\lfloor \frac{n}{m} \rfloor + 1$ partitions of n modulo m.

Case 3: m is odd.

Now $2a \equiv 2b \equiv n \pmod{m}$ implies that $a \equiv b \equiv 2^{-1}n \equiv \frac{m+1}{2}n \pmod{m}$. Let $r \in \{0, 1, 2, \ldots, m-1\}$ such that $r \equiv \frac{m+1}{2}n \pmod{m}$. Then any partition of n modulo m into 2 parts has the form $(r + im) + (n - (r + im))$ for $0 \leq i \leq \frac{1}{m}\left(\frac{n}{2} - r\right)$. Therefore there are $\lfloor \frac{1}{m}\left(\frac{n}{2} - r\right) \rfloor + 1$ such partitions. ■

Corollary 5 *Given integer $m \geq 2$, the sequence $P_n = P_{2,m}(n)$ has the following recursion relations:*

1. *If m is odd, $P_n = P_{n-2m} + 1$.*
2. *If m is even, $P_{2n} = P_{2n-m} + 1$.*
3. *$P_n = P_{n-2} + \chi_0(n)$, where χ_0 is the characteristic function on the congruence class of 0 modulo m, $\chi_0(n) = \begin{cases} 1 & n \equiv 0 \mod m \\ 0 & \text{otherwise.} \end{cases}$*

Proof. The first two relations can be derived from the formula formula for $P_{2,m}(n)$ by induction. We will only prove the third, as it is the most general.

Suppose $a_1 + b_1, a_2 + b_2, \ldots, a_{P_n} + b_{P_n}$ are the 2-part partitions of n modulo m. Then $(a_1 + 1) + (b_1 + 1), (a_2 + 1) + (b_2 + 1), \ldots, (a_{P_n} + 1) + (b_{P_n} + 1)$ are each partitions of $n + 2$ modulo m. Additionally, if $n \equiv 0 \pmod{m}$ then $0 + n$ is also a partition of n modulo m.

To be sure that all partitions of n modulo m have been counted, suppose $a + b$ is a partition of n modulo m, where $a > 0$. Then $(a - 1) + (b - 1)$ is a partition of $n - 2$ modulo m, so $a - 1 = a_i$ and $b - 1 = b_i$ for some $i \in \{1, 2, \ldots, P_{n-2}\}$. Thus $P_n = P_{n-2} + \chi_0(n)$. ■

Thus we have found a recursion relation and an explicit formula for $k = 2$ and a general m. Interestingly enough, when $m = 3$, a surprisingly simple explicit formula exists, which we present without proof.

$$P_{2,3}(n) = \left\lfloor \frac{n+2}{2} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor, \tag{5}$$

4 Special Case: m=k

We next consider the special case of $P_{k,m}(n)$ where $m = k$. That is, we are partitioning n into k parts that are congruent to one another modulo k. First we prove that we need only explore multiples of k as values for n.

Theorem 6 *If n is a positive integer not a multiple of k, then $P_{k,k}(n) = 0$.*

Proof. Suppose $n = a_1 + a_2 + \ldots + a_k$ such that $a_1 \equiv a_2 \equiv \ldots \equiv a_k \equiv r \mod k$. Define b_i such that $b_i k + r = a_i$. The we have

$$n = \sum_{i=1}^{k} a_i = \sum_{i=1}^{k} (b_i k + r) = \left(\sum_{i=1}^{k} b_i \right) k + rk = \left(\sum_{i=1}^{k} b_i + r \right) k \tag{6}$$

So $n \equiv 0 \mod k$, which contradicts our hypothesis that n is not a multiple of k. Therefore $P_{k,k}(n) = 0$ since no such partition could exist. ■

For simplicity we now define $P_k(n) = P_{k,k}(kn)$. The generating function for $P_k(n)$ becomes a bit simpler as we strip out the zero coefficients.

$$f(x) = \frac{1}{(1-x)^2(1-x^2)(1-x^3)\ldots(1-x^{k-1})} \tag{7}$$

This is the generating function for partitions of n using only $1, 2, \ldots, k-1$ and an alternate 1 (denoted here as $1'$) as parts.

Through this new interpretation of $P_k(n)$, we develop a recurrence relation for $P_k(n)$.

Theorem 7 *For $k \geq 2$ and $n \geq 0$, $P_k(n) = P_{k-1}(n) + P_k(n-(k-1))$*

Proof. We will interpret $P_k(n)$ as the number of partitions of n into $1, 2, \ldots, k-1$, and $1'$. These partitions can be split into two families: those that contain $k-1$ as a part, and those that do not.

There are $P_k(n-(k-1))$ partitions of n that contain $k-1$. This is apparent since removing $k-1$ from each of these partitions makes it a partition of $n-(k-1)$.

Any partition of n that does not use $k-1$ or anything higher is equivalently a partition of n into $1, 2, \ldots, k-2$ and $1'$. Thus there are $P_{k-1}(n)$ partitions of n that do not contain $k-1$.

Together we get $P_{k-1}(n)+P_k(n-(k-1))$ partitions of n using $1, 2, \ldots, k-1$ and $1'$. ■

Starting with knowledge that $P_1(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$, and letting $P_k(n) = 0$ when $n < 0$, we generate as many values of $P_k(n)$ as desired by using the recurrence relation.

$n \backslash k$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	1	2	2	2	2	2	2	2	2	2
2	1	3	4	4	4	4	4	4	4	4
3	1	4	6	7	7	7	7	7	7	7
4	1	5	9	11	12	12	12	12	12	12
5	1	6	12	16	18	19	19	19	19	19
6	1	7	16	23	27	29	30	30	30	30
7	1	8	20	31	38	42	44	45	45	45
8	1	9	25	41	53	60	64	66	67	67
9	1	10	30	53	71	83	90	94	96	97
10	1	11	36	67	94	113	125	132	136	138
11	1	12	42	83	121	150	169	181	188	192
12	1	13	49	102	155	197	227	246	258	265
13	1	14	56	123	194	254	298	328	347	359
14	1	15	64	147	241	324	388	433	463	482
15	1	16	72	174	295	408	498	564	609	639
16	1	17	81	204	359	509	634	728	795	840
17	1	18	90	237	431	628	797	929	1025	1092
18	1	19	100	274	515	769	996	1177	1313	1410
19	1	20	110	314	609	933	1231	1477	1665	1803
20	1	21	121	358	717	1125	1513	1841	2099	2291

Table 1: Sample values for $P_k(n)$

Fixing n and using k as a parameter (i.e. reading across the rows of Table 1), we see that the sequence generated by $P_k(n)$ is eventually constant, evening out when $k > n$. This is logical since $P_k(n)$ counts the number of partitions of n using $1', 1, 2, \ldots, k-1$ as parts. When $k > n$, there are more "allowable" parts, but the parts themselves are greater than n and so cannot be used to partition n. Thus we have the following corollary:

Corollary 8 *$P_k(n)$ is equal to the total number of partitions of n using two types of 1 when $n < k$.*

One last aspect to examine is the existence of an explicit formula for $P_k(n)$ for nonnegative values of n. Unfortunately no formula for an arbitrary k is yet to be found. The formula for a specific value of k can be derived by decomposing the generating function by partial fractions and combining power series. In this way we have derived the formulae for $k = 1, 2, 3, 4$ and 5.

$P_1(n) = 1$

$P_2(n) = n + 1$

$P_3(n) = \frac{n^2}{4} + n + \frac{7+(-1)^n}{8}$

$$P_4(n) = \frac{n^3}{36} + \frac{7n^2}{24} + \frac{11n}{12} + \frac{103}{144} + \frac{(-1)^n}{16} + \frac{1}{9}\begin{cases} 1 & n \equiv 0 \mod 3 \\ 0 & n \equiv 1 \mod 3 \\ -1 & n \equiv 2 \mod 3 \end{cases}$$

$$P_5(n) = \frac{n^4}{576} + \frac{11n^3}{288} + \frac{83n^2}{288} + \frac{55+(-1)^n}{64}n + \frac{2815}{3456} + \frac{(-1)^n 11}{128} + \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{16} + \frac{1}{27}\begin{cases} -2 & n \equiv 2 \mod 3 \\ 1 & otherwise \end{cases}$$

Clearly the formulas threaten to get very complicated very quickly as k increases.

5 Special Case: $m = lk$

We new consider when m is a multiple of k. Our first observation is that the results of Theorem 6 given above generalizes to $m = lk$ for any $l \in \mathbb{N}$.

Theorem 9 *If n is a positive integer not a multiple of k, then $P_{k,lk}(n) = 0$ for any natural number l.*

The proof is fundamentally unchanged and is left to the reader.

The further connections to the case explored in the previous section are perhaps more surprising.

Theorem 10 *For any $k, l \in \mathbb{N}$ and nonnegative integer n, $P_{k,lk}(kn) = P_{k,k}\left(k\left\lfloor \frac{n}{l} \right\rfloor\right)$.*

In other words, the sequence created by $P_{k,lk}(n)$ as $n = 1, 2, 3, \ldots$ is the sequence created by $P_{k,k}(n)$ as $n = 1, 2, 3, \ldots$ where each term in the sequence is repeated l times.

Proof. We first look at the generating function for $P_{k,lk}(n)$. Stripping out the zero terms of the sequence as in section 4, we are left with the generating function

$$\frac{1}{(1-x)(1-x^l)(1-x^{2l})\cdots(1-x^{(k-1)l})}. \tag{8}$$

Thus $P_{k,lk}(kn)$ can be interpreted as the number of ways to partition n using $1, l, 2l, \ldots, (k-1)l$ as parts, just as $P_{k,k}\left(k\left\lfloor \frac{n}{l}\right\rfloor\right)$ can be interpreted as the number of ways to partition $\left\lfloor \frac{n}{l}\right\rfloor$ using $1', 1, 2, \ldots, k-1$ as parts. It remains to show that these two quantities are equal for any n.

By the division algorithm, $n = pl + q$ for some integers p and q. Note that $p = \left\lfloor \frac{n}{l}\right\rfloor$ and $q < l$. Obviously there is exactly one way to partition q into the sum of ones. Furthermore, for every way to partition pl into $1, l, 2l, \ldots, (k-1)l$, you can partition p into $1', 1, 2, \ldots, (k-1)l$ by substituting 1 for 1', l for 1, $2l$ for 2, and so on. Therefore $P_{k,lk}(kn) = P_{k,k}\left(k\left\lfloor \frac{n}{l}\right\rfloor\right)$. ■

6 Conclusion

We have explored many special cases of $P_{k,m}(n)$, including finding explicit formulas for $P_{2,m}(n)$, and $P_{k,k}(kn)$ for $k = 1, 2, 3, 4$ and 5. The most useful result for future research is the first theorem, though, as knowing the generating function accelerates the study of sequences immensely.

Clearly this problem is far from completely solved, however. No explicit formula for an arbitrary k, m, and n exist yet, and many more special cases to explore remain. Section 5 only scratches the surface of what effect common factors between m and k could have. Most likely we could study the problem specifically for $k = 3, 4, \ldots$ as in Section 3, although that could turn out to be a tedious process to say the least.